Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 2146–2157
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Homogeneous instanton bundles on $\mathbb{P}^3$ for the action of $\mathsf{SL}(2)$

Daniele Faenzi *,1

Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/a, I-50134, Florence, Italy

Received 3 November 2006; received in revised form 13 June 2007; accepted 24 June 2007
Available online 29 June 2007

Abstract

We classify k-instanton bundles on $\mathbb{P}^3_{\mathbb{C}}$ which are homogeneous for the group $\mathsf{SL}(2)$, acting linearly on $\mathbb{P}^3$ with an open orbit. Besides the classical special instantons, we find isolated examples for $\mathsf{SL}(2)$ acting by the representation of binary cubics. We show that these examples are unique and that they exist only for $k = a(a-1)/2$, for some $a \geq 2$. We also compute their minimal free resolution in terms of homogeneous equivariant matrices.

MSC: primary 14J60; secondary 14L30; 14F05

Keywords: Instanton bundles; Group action on moduli spaces; Special instantons; Binary cubics; Equivariant resolutions; Quasi-homogeneous spaces; Representations of $\mathsf{SL}(2)$

1. Introduction and preliminaries

A k-instanton bundle on the complex projective space $\mathbb{P}^3 = \mathbb{P}(V)$ is a rank-2 stable bundle $\mathscr{E}$ with $c_1(\mathscr{E}) = 0$, $c_2(\mathscr{E}) = k$, $\mathrm{H}^1(\mathbb{P}^3, \mathscr{E}(-2)) = 0$; see [8,2]. According to the ADHM correspondence introduced in [1], instantons satisfying a reality condition can be seen in terms of self-dual Yang–Mills $\mathsf{Sp}(1)$-connections on S^4. The moduli space of k-instantons will be denoted by $\mathsf{MI}(k)$. It is conjecturally smooth and irreducible, and proved to be so up to $k = 5$; see [11,3].

Assume now that a simple complex Lie group G acts on V via a representation $\rho : \mathsf{G} \to \mathsf{SL}(V)$, and consider the G-action induced on $\mathsf{MI}(k)$ by pull-back. We will be interested in the *fixed points* in $\mathsf{MI}(k)$ for this action, namely G-*homogeneous instanton bundles*.

We suppose that the group G acts on the space $\mathbb{P}^3 = \mathbb{P}(V)$ with an open orbit, i.e. $\mathbb{P}^3$ is a quasi-homogeneous G-space. Then either G acts transitively (and in this case, up to a finite cover, G is isomorphic to $\mathsf{SL}(4)$ or to $\mathsf{Sp}(2)$), or G must be isomorphic to $\mathsf{SL}(2)$ up to a finite cover.

For $\mathsf{SL}(4)$, no homogeneous instanton bundle exists. In the case $\mathsf{G} = \mathsf{Sp}(2)$, the bundle $\mathscr{E}$ must be isomorphic to a null-correlation bundle. So we assume $\mathsf{G} \cong \mathsf{SL}(2)$, and the action is given by a decomposition of V into $\mathsf{SL}(2)$-modules. We denote by U the standard representation of $\mathsf{SL}(2)$ and by U_b the module $\mathrm{Sym}^b U$. Then the decomposition of V must be one of the following types:

* Tel.: +39 0554237139; fax: +39 0554222695.
E-mail address: faenzi@math.unifi.it.
URL: http://www.math.unifi.it/~faenzi/.

1 Author partially supported by MIUR and GNSAGA-INDAM.

0393-0440/$ - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.geomphys.2007.06.003

(A) the *special action*: $V \cong U \oplus U$;
(B) the representation of *binary cubics*: $V \cong U_3$.

According to the above cases, we prove here the following result.

Theorem. *Let* $\mathsf{SL}(2)$ *act linearly with an open orbit on* $\mathbb{P}^3 = \mathbb{P}(V)$ *and let* $\mathscr{E}$ *be an* $\mathsf{SL}(2)$*-homogeneous instanton bundle on* $\mathbb{P}(V)$.

(A) *If the* $\mathsf{SL}(2)$*-module* V *is isomorphic to* $U \oplus U$, *then* $\mathscr{E}$ *is a special instanton.*
(B) *If the* $\mathsf{SL}(2)$*-module* V *is isomorphic to* U_3, *then* $c_2(\mathscr{E}) = \binom{d+1}{2}$ *for some* $d \geq 1$, *and the bundle* $\mathscr{E}$ *is unique up to isomorphism.*

Moreover, for any integer $d \geq 1$ *there exists a unique minimal equivariant exact sequence of the form*

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-2d-1) \to U_{3d} \otimes \mathscr{O}_{\mathbb{P}^3}(-d-1) \to U_{3d+1} \otimes \mathscr{O}_{\mathbb{P}^3}(-d) \to \mathscr{E} \to 0,$$

which defines the unique $\mathsf{SL}(2)$*-homogeneous instanton* $\mathscr{E}$ *of case* (B), *with* $c_2(\mathscr{E}) = \binom{d+1}{2}$.

The paper consists of two parts. In the first one we briefly consider the case (A), where we reduce to the set-up already studied in the literature, namely the special instantons. In the second part, we study the case (B), and we provide new examples of homogeneous instantons. These bundles are first studied via the classical monad-theoretic approach, then constructed in a simpler way via their minimal graded free equivariant resolution. We prove also that these examples are unique.

We will work on a complex projective variety Y, embedded by $\mathscr{O}_Y(1)$. Given a sheaf $\mathscr{F}$ on Y we write $\mathscr{F}(t)$ for $\mathscr{F} \otimes \mathscr{O}_Y(1)^{\otimes t}$. Sometimes we will deal with products $Y = Y_1 \times \cdots \times Y_r$, embedded by $\mathscr{O}_Y(1, \ldots, 1)$, with obvious notation.

We say that a group G acts linearly on a projective variety $Y \subset \mathbb{P}(V)$ if G can be identified with a subgroup of $\mathsf{GL}(V)$ that takes Y to Y. If Y is a product of projective spaces, then G acts *separately* if it acts linearly on Y under the Segre embedding. A sheaf $\mathscr{F}$ on Y is called G-homogeneous if we have $\mathscr{F} \cong \phi^*(\mathscr{F})$, for all transformations $\mathsf{G} \ni \phi : Y \to Y$.

Given a vector bundle $\mathscr{E}$ on $\mathbb{P}^3$, we will set $I = \mathrm{H}^1(\mathbb{P}^3, \mathscr{E}(-1))$, $W = \mathrm{H}^1(\mathbb{P}^3, \mathscr{E} \otimes \Omega)$. An instanton bundle $\mathscr{E}$ on $\mathbb{P}^3$ is a stable rank-2 bundle with $c_1(\mathscr{E}) = 0$, $c_2(\mathscr{E}) = k$, which is isomorphic to the cohomology of a monad of the following form:

$$I^* \otimes \mathscr{O}_{\mathbb{P}^3}(-1) \xrightarrow{JA^\top} W \otimes \mathscr{O}_{\mathbb{P}^3} \xrightarrow{A} I \otimes \mathscr{O}_{\mathbb{P}^3}(1),$$

where $\dim(I) = k$, $\dim(W) = 2k+2$, and $J : W^* \to W$ is a skew-symmetric duality. This is equivalent to the definition that we have given before; see for instance [2,12].

According to [5], the moduli space of instanton bundles $\mathsf{MI}(k)$ can be defined as the GIT quotient:

$$\{A \in \mathrm{Hom}_{\mathbb{P}^3}(W \otimes \mathscr{O}_{\mathbb{P}^3}, I \otimes \mathscr{O}_{\mathbb{P}^3}(1))^\circ | AJA^\top = 0\} // \mathsf{Sp}(W) \times \mathsf{GL}(I),$$

where $\mathrm{Hom}_{\mathbb{P}^3}(W \otimes \mathscr{O}_{\mathbb{P}^3}, I \otimes \mathscr{O}_{\mathbb{P}^3}(1))^\circ$ is the open complement in $\mathrm{Hom}_{\mathbb{P}^3}(W \otimes \mathscr{O}_{\mathbb{P}^3}, I \otimes \mathscr{O}_{\mathbb{P}^3}(1))$ of a hypersurface $\mathscr{V}$, and the group $\mathsf{Sp}(W) \times \mathsf{GL}(I)$ acts via the standard left and right multiplication. The homogeneous form corresponding to $\mathscr{V}$ is $\mathsf{Sp}(W) \times \mathsf{SL}(I) \times \mathsf{SL}(V)$-invariant, and associates with $A \in \mathrm{Hom}_{\mathbb{P}^3}(W \otimes \mathscr{O}_{\mathbb{P}^3}, I \otimes \mathscr{O}_{\mathbb{P}^3}(1))$ the determinant of the induced map $W \otimes I \to \mathrm{Sym}^2 I \otimes V$.

2. Special action

We call *special action* the $\mathsf{SL}(2)$-action on V by $V \cong U \oplus U$. Indeed, $\mathsf{SL}(2)$ acts on V this way if $\mathscr{E}$ is a *special* instanton bundle; see [4]. Special instantons were first studied in [10], and have been extensively investigated ever since, see e.g. [14,13], so we will say no more about them here.

However, an $\mathsf{SL}(2)$-homogeneous instanton bundle for the special action need not a priori be a special instanton. We show here that this is indeed the case. In fact setting $k = c_2(\mathscr{E})$, by [4, Proposition 4.12], it suffices to show the isomorphisms of representations:

$$I \cong U_0^k, \qquad W \cong U^{k+1}. \tag{2.1}$$

We need the following lemma, which is essentially due to Vallès, see [15]. We sketch a proof for the reader's convenience.

Lemma 2.1 (Vallès). *Let $\mathscr{F}$ be a vector bundle with $c_1(\mathscr{F}) = 0$, defined on $Q = \mathbb{P}^1 \times \mathbb{P}^1$, and let* $\mathsf{SL}(2)$ *act separately on Q, transitively on the second factor. Assume that $\mathscr{F}$ is* $\mathsf{SL}(2)$*-equivariant. Then $\mathscr{F}$ is an extension of line bundles.*

Proof. Let $p : Q \to \mathbb{P}^1$ be the $\mathsf{SL}(2)$-equivariant projection onto the second factor and consider $\mathscr{G} = p_*(\mathscr{F})$. The vector bundle $\mathscr{G}$ is $\mathsf{SL}(2)$-homogeneous for the induced action on $\mathbb{P}^1$, and it decomposes as the direct sum of two line bundles by Grothendieck's theorem. Given an integer a define the subset

$$Z_a = \{y \in \mathbb{P}^1 \mid \mathscr{F}_{|\mathbb{P}^1 \times \{y\}} \simeq \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(-a)\},$$

and take $\overline{a}$ as the minimal nonnegative integer a such that Z_a is nonempty. Of course $Z_{\overline{a}}$ is open in $\mathbb{P}^1$. However $Z_{\overline{a}}$ must contain an orbit for the action of $\mathsf{SL}(2)$; hence it is all of $\mathbb{P}^1$ by our assumption. Therefore $p_*(\mathscr{F}(-a, -a))$ is isomorphic to a line bundle $\mathscr{O}_{\mathbb{P}^1}(b)$ and we have a natural epimorphism $\psi : \mathscr{F}^*(a, a) \to \mathscr{O}_Q(0, -b)$. Indeed, for each y, ψ restricts over $\mathbb{P}^1 \times \{y\}$ to the projection onto the second factor: $\mathscr{F}^*(a)_{|\mathbb{P}^1 \times \{y\}} \cong \mathscr{O}_{\mathbb{P}^1}(2a) \oplus \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}$. So our claim is proved. □

Lemma 2.2. *Let $\mathscr{F}$ be as above and assume $\mathrm{H}^0(Q, \mathscr{F}) = 0$.*

(i) *If* $\mathsf{SL}(2)$ *acts transitively on the first factor, then we have an exact sequence:*

$$0 \to \mathscr{O}_Q(a, -a-1) \to \mathscr{F} \to \mathscr{O}_Q(-a, a+1) \to 0, \quad \text{for some } a \in \mathbb{Z}, \tag{2.2}$$

and the bundle $\mathscr{F}$ is unique up to isomorphism.

(ii) *If* $\mathsf{SL}(2)$ *acts trivially on the first factor, then we have an exact sequence:*

$$0 \to \mathscr{O}_Q(a, -1) \to \mathscr{F} \to \mathscr{O}_Q(-a, 1) \to 0, \quad \text{for some } a > 0. \tag{2.3}$$

Proof. We know by the previous lemma that $\mathscr{F}$ is an extension of line bundles of the form

$$0 \to \mathscr{O}_Q(a, b) \to \mathscr{F} \to \mathscr{O}_Q(-a, -b) \to 0, \quad \text{for some } a, b \in \mathbb{Z}.$$

Since $\mathrm{H}^0(Q, \mathscr{F}) = 0$, this exact sequence must be nontrivial, so the group $\mathrm{H}^1(Q, \mathscr{O}_Q(2a, 2b))$ must be nonzero. More than that, it must contain a nonzero element which is invariant for $\mathsf{SL}(2)$. We have two possibilities: either $a \geq 0, b \leq -1$ or $a \leq -1, b \geq 0$. Take the first case (the other one is analogous). According to the alternatives (i) or (ii), the $\mathsf{SL}(2)$ module $\mathrm{H}^1(Q, \mathscr{O}_Q(2a, 2b))$ is isomorphic to $U_{2a} \otimes U_{-2-2b}$ or to $\mathbb{C}^{2a+1} \otimes U_{-2-2b}$. In the former case this module contains a nonzero invariant element if and only if $b = a - 1$, and in this case the extension is unique. In the latter case we have $b = -1$, and we get $a > 0$ by $\mathrm{H}^0(Q, \mathscr{F}) = 0$. Notice that there is a $(2a + 1)$-dimensional vector space of invariant elements in this case. □

Lemma 2.3. *Let $\mathscr{E}$ be an* $\mathsf{SL}(2)$*-homogeneous vector bundle on $\mathbb{P}(U \oplus U)$. Then (2.1) takes place.*

Proof. Let us fix isomorphisms $V \cong U \oplus U \cong U \otimes U'$, where U' is a two-dimensional vector space where $\mathsf{SL}(2)$ acts trivially. Then $\mathbb{P}(U \oplus U)$ contains an invariant smooth quadric $Q \cong \mathbb{P}(U) \times \mathbb{P}(U')$. Denote by $\mathscr{F}$ the restriction of $\mathscr{E}$ to Q, and observe that $\mathscr{F}$ is $\mathsf{SL}(2)$-homogeneous. We have the exact sequence

$$0 \to \mathscr{E}(-3) \to \mathscr{E}(-1) \to \mathscr{F}(-1) \to 0. \tag{2.4}$$

The vanishing $\mathrm{H}^1(\mathbb{P}^3, \mathscr{E}(-2)) = 0$ implies $\mathrm{H}^0(Q, \mathscr{F}) = 0$. Thus we can apply Lemma 2.2, part (2.3), to $\mathscr{F}$. It follows at once that

$$\mathrm{H}^1(Q, \mathscr{F}(-1)) \cong U_0^a, \quad \text{for some } a.$$

On the other hand by (2.4) we get $I \oplus I^* \cong U_0^{2a}$, and so $I \cong U_0^a$, $a = k$. To read the structure of W we use the isomorphism provided by [5]. This gives

$$W \otimes I \cong \mathrm{Sym}^2 I \otimes V \cong U^{k(k+1)}.$$

We obtain $W \cong U^{k+1}$. □

3. Action by binary cubics

Here we consider the action of $\mathsf{SL}(2)$ over the vector space V by identifying V with the space U_3 of binary cubics, over the two-dimensional vector space U generated by x and y. We let x_0, x_1, x_2, x_3 have weights of, respectively, 3, 1, -1, -3 for the action of $\mathfrak{sl}(2)$. In other words we think of the identities

$$x_0 = \mathsf{x}^3, \qquad x_1 = \mathsf{x}^2\mathsf{y}, \qquad x_2 = \mathsf{x}\mathsf{y}^2, \qquad x_3 = \mathsf{y}^3.$$

The space $\mathbb{P}(U_3)$ is decomposed into three orbits of dimensions 1, 2, 3. The one-dimensional orbit is a twisted cubic Γ, which sits as the singular locus of the quartic surface Y_4 defined by the discriminant equation F_4:

$$F_4 = x_0^2x_3^2 - 4x_0x_2^3 + 6x_0x_1x_2x_3 - 4x_1^3x_3 + x_1^2x_2^2.$$

In turn the ideal of Γ is given by the Jacobian of F_4. Though rather trivial, the following lemma is often useful.

Lemma 3.1. *Let $\psi : \mathcal{A} \to \mathcal{B}$ be an equivariant morphism of $\mathsf{SL}(2)$-homogeneous vector bundles on $\mathbb{P}(U_3)$.*

– *If $\mathrm{rk}(\psi)_{p_0} = r$ for some $p_0 \in \Gamma$, then $\mathrm{rk}(\psi) \geq r$ everywhere.*
– *If $\mathrm{rk}(\psi)_{p_1} = r$ for some $p_1 \in \mathbb{P}(U_3) \setminus Y_4$, then $\mathrm{rk}(\psi) \leq r$ everywhere.*

Proof. Consider the subsets

$$\{q \in \mathbb{P}(U_3) \mid \mathrm{rk}(\psi)_q \leq r - 1\},$$
$$\{q \in \mathbb{P}(U_3) \mid \mathrm{rk}(\psi)_q > r\}.$$

The first is a closed $\mathsf{SL}(2)$-invariant subset of $\mathbb{P}(U_3)$. Thus it contains all of Γ as soon as it is nonempty. But it does not contain p_0, so it must be empty. The second is an open $\mathsf{SL}(2)$-orbit, so it should contain all of $\mathbb{P}(U_3) \setminus Y_4$. But it does not contain p_1, so it must be empty. □

3.1. *The $\mathsf{SL}(2)$-structure of I and W*

There is a natural equivariant 6 : 1 branched cover $\mathbb{P}(U)^{\times 3} \to \mathbb{P}(U_3)$, associated with the embedding $U_3 \hookrightarrow U^{\otimes 3}$. This covering factorizes as $\mathbb{P}(U)^{\times 3} \xrightarrow{2:1} \mathbb{P}(U) \times \mathbb{P}(U_2) \xrightarrow{3:1} \mathbb{P}(U_3)$. In the following lemma we establish some properties of these maps.

Lemma 3.2. *We have a commutative diagram of $\mathsf{SL}(2)$-equivariant maps:*

$$\begin{array}{ccccccc}
\mathbb{P}(U) & \overset{\alpha}{\hookrightarrow} & \mathbb{P}(U) \times \mathbb{P}(U) & \overset{\beta}{\hookrightarrow} & \mathbb{P}(U) \times \mathbb{P}(U_2) & \overset{\gamma}{\hookrightarrow} & \mathbb{P}(U_3 \oplus U_1) \\
\cong \downarrow & & \downarrow & & 3:1 \downarrow f & \overset{\pi}{\swarrow} & \\
\Gamma & \longrightarrow & Y_4 & \longrightarrow & \mathbb{P}(U_3), & &
\end{array}$$

where π is defined by $U_3 \hookrightarrow U_3 \oplus U_1$, $\Gamma \hookrightarrow Y_4 \hookrightarrow \mathbb{P}(U_3)$ are the natural embeddings, $\alpha = \varphi_{|\mathcal{O}_{\mathbb{P}^1}(2)|}$, $\beta = \varphi_{|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2)|}$, $\gamma = \varphi_{|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,1)|}$. We have the equivariant isomorphisms

$$f_*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}) \cong \mathcal{O}_{\mathbb{P}^3} \oplus U \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \tag{3.1}$$
$$f_*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(-2, 0)) \cong \Omega_{\mathbb{P}^3}, \tag{3.2}$$
$$f_*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0, -1)) \cong U \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2), \tag{3.3}$$
$$f_*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0, -2)) \cong U_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-2). \tag{3.4}$$

Proof. Denote by h_1, h_2, h the very ample tautological divisors respectively on $\mathbb{P}(U)$, $\mathbb{P}(U_2)$, $\mathbb{P}(U_3)$. It is clear that all the maps in the above diagram are equivariant. The map f is evidently a triple cover, and we have $f^*(\mathcal{O}_{\mathbb{P}^3}(1)) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 1)$. The ramification divisor of f has degree 4, since $(2h_1 + h_2) \cdot (h_1 + h_2)^2 = 4$. It corresponds to the unique invariant element of $\mathrm{H}^0(\mathbb{P}(U) \times \mathbb{P}(U_2), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2, 1))$. We have $f_*(h_1) = h$, $f_*(h_2) = 2h$. The image of β is the tangential variety of $\mathrm{Im}(\beta \circ \alpha)$. It corresponds to the unique invariant element of $\mathrm{H}^0(\mathbb{P}(U) \times \mathbb{P}(U_2), \mathscr{O}(0,2))$; it also has degree 4.

On $\mathbb{P}(U)$ we have $\alpha^*\beta^* f^*(\mathscr{O}_{\mathbb{P}^3}(1)) \cong \alpha^*\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2) \cong \mathscr{O}_{\mathbb{P}^1}(3)$, so the rational curves Γ and $f(\beta(\alpha(\mathbb{P}(U))))$ are identified. This identification obviously extends to the tangential divisors so the diagram commutes. Notice that the map $\mathbb{P}(U) \times \mathbb{P}(U) \to Y_4$ is nothing but the embedded resolution of singularities of Y_4.

Observe that $f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2})$ is a rank-3 vector bundle. This bundle splits by the projection formula and Horrocks's criterion. Of course it contains $\mathscr{O}_{\mathbb{P}^2}$ as a direct summand. We have $\mathsf{SL}(2)$-isomorphisms:

$$U_3 \oplus U \cong \mathrm{H}^0(\mathbb{P}(U) \times \mathbb{P}(U_2), \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,1)) \cong \mathrm{H}^0(\mathbb{P}^3, f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2})(1)).$$

We conclude that the remaining summand of $f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2})$ is isomorphic to $U(-1)$. One treats similarly $f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,-1))$ and $f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,-2))$. For $f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-2,0))$, notice that $\mathrm{H}^1(\mathbb{P}^3, f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-2,0))(t)) = 0$ for $t \neq 0$, $\mathrm{h}^1(\mathbb{P}^3, f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-2,0))) = 1$, which implies $f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-2,0)) \cong \Omega$. We have thus proved (3.1)–(3.4). □

Setting $Q = \mathrm{Im}(\beta)$, we can write the equivariant exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,-2) \to \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2} \to \mathscr{O}_Q \to 0. \tag{3.5}$$

Lemma 3.3. *Let $\mathscr{E}$ be an $\mathsf{SL}(2)$-equivariant instanton bundle on $\mathbb{P}(U_3)$. Then we have isomorphisms of $\mathsf{SL}(2)$-modules:*

$$I \cong \wedge^2 U_a \qquad W \cong U_{2a+1} \oplus U_{a-2} \otimes U_{a-1},$$

for some $a \geq 1$; in particular we have $c_2(\mathscr{E}) = \binom{a}{2}$.

Proof. Recall the map f of Lemma 3.2 and set $\mathscr{K} = f^*(\mathscr{E})$. The bundle $\mathscr{K}$ is clearly $\mathsf{SL}(2)$-homogeneous, and notice that it restricts to a homogeneous bundle $\mathscr{F}$ on the invariant divisor Q. Therefore $\mathscr{F}$ satisfies the hypothesis of Lemma 2.2, part (i), indeed the action is transitive on both factors of $Q \cong \mathbb{P}(U) \times \mathbb{P}(U)$.

Using the projection formula, (3.1) and the condition $\mathrm{H}^2(\mathbb{P}^3, \mathscr{E}(-2)) = 0$, we can write the equivariant isomorphisms:

$$\begin{aligned} I &= \mathrm{H}^1(\mathbb{P}^3, \mathscr{E}(-1)) \oplus \mathrm{H}^1(\mathbb{P}^3, \mathscr{E} \otimes U(-2)) \\ &\cong \mathrm{H}^1(\mathbb{P}^3, \mathscr{E} \otimes f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-1,-1))) \\ &\cong \mathrm{H}^1(\mathbb{P}(U) \times \mathbb{P}(U_2), \mathscr{K}(-1,-1)). \end{aligned}$$

Tensoring with $\mathscr{K}(-1,-1)$ the exact sequence (3.5) we obtain

$$0 \to \mathscr{K}(-1,-3) \to \mathscr{K}(-1,-1) \to \mathscr{F}(-1,-2) \to 0.$$

Making use of (3.4) and Serre duality, we get the equivariant isomorphism:

$$\mathrm{H}^2(\mathbb{P}(U) \times \mathbb{P}(U_2), \mathscr{K}(-1,-3)) \cong I^* \otimes U_2.$$

The computation of $\mathrm{H}^1(\mathbb{P}^3, \mathscr{F}(-1,-2))$ is carried out using (2.2), and yields the exact sequence

$$0 \to I \to U_{a-1} \otimes U_{a+1} \oplus U_{a-1} \otimes U_{a-1} \to I^* \otimes U_2 \to 0,$$

with $a \geq 1$. It follows that the decomposition of I contains the summand U_{2a-2} with multiplicity 1. Ordering the summands of this decomposition by decreasing weight, the next term must then be U_{2a-6} (for $a \geq 3$) and so forth, one proves inductively that I is isomorphic to $\wedge^2 U_a$.

Let us consider W. Again by (3.2) we have the isomorphism

$$W = \mathrm{H}^1(\mathbb{P}^3, \mathscr{E} \otimes f_*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-2,0))) \cong \mathrm{H}^1(\mathbb{P}(U) \times \mathbb{P}(U_2), \mathscr{K}(-2,0)).$$

Tensoring (3.5) by $\mathscr{K}(-2,0)$, substituting the expressions of I and of $\mathrm{H}^1(\mathbb{P}(U) \times \mathbb{P}(U_2), \mathscr{F}(-2,0))$, and using our cohomology vanishing, we arrive at the equivariant exact sequence:

$$0 \to W \to U_{a-1} \otimes U_{a-2} \oplus U_{a+1} \otimes U_a \to (\wedge^2 U_a) \otimes U \to 0.$$

The rightmost term is isomorphic to $U_a \otimes U_{a-1}$. Notice that once we remove the summand U_{2a+1} from the middle term, it becomes isomorphic to $U_a \otimes U_{a-1} \oplus U_{a-1} \otimes U_{a-2}$. This proves our claim. $\square$

3.2. Equivariant matrices

We will consider the standard basis of U_b defined by $y_k = \mathsf{x}^{b-k}\mathsf{y}^k$, with the natural induced action of $\mathsf{SL}(2)$, or equivalently of $\mathfrak{sl}(2)$. As usual, the Lie algebra $\mathfrak{sl}(2)$ will be generated by $\mathsf{X}, \mathsf{Y}, \mathsf{H}$ with $[\mathsf{H}, \mathsf{X}] = 2\mathsf{X}$, $[\mathsf{H}, \mathsf{Y}] = -2\mathsf{Y}$, $[\mathsf{X}, \mathsf{Y}] = \mathsf{H}$. This leads us to adopt the convention of Y acting on $y_k \in U_b$ by

$$\mathsf{Y} \cdot y_k = (b-k) y_{k+1}.$$

We extend this convention to the tensor algebra by linearity. This gives a uniform way of writing down the coefficients of the $\mathfrak{sl}(2)$-action on tensor products and homomorphism spaces. In particular, denoting by y_q (resp. by z_p) the basis vectors of U_b (resp. of U_c), the following formula gives the $\mathfrak{sl}(2)$-action on the maximal weight vector w of the summand U_{b+c-2s} of $U_b \otimes U_c$:

$$\mathsf{Y}^j(w) = \underbrace{\mathsf{Y} \cdots \mathsf{Y} \cdot \mathsf{Y}}_{j} \cdot w = \frac{\sum_{p=0}^{j+s} \sum_{q=0}^{s} (-1)^q \binom{s}{q} \binom{b-q}{p-q} \binom{c-s+q}{j-p+q}}{\binom{b+c-2s}{j}} y_{j-p+s} \otimes z_p.$$

For instance the generator of $U_4 \subset U_3 \otimes U_3$ with this convention is $x_0^2 x_3^2 - 4x_0 x_2^3 + 6x_0 x_1 x_2 x_3 - 4x_1^3 x_3 + x_1^2 x_2^2$, the tangential quartic to the twisted cubic, which we have used before.

Remark 3.4. Given an integer $b \geq 3$, there are only four integers c such that there exists an equivariant linear map $g : U_b \to U_c \otimes \mathscr{O}_{\mathbb{P}^3}(1)$, namely $c \in \{b+3, b+1, b-1, b-3\}$ and of course if b equals 0, 1 or 2 there are respectively 2, 3, 3 choices. Anyway the map g is unique up to a nonzero scalar. We will adopt the notation

$$f_b^{++} : U_b \to U_{b-3} \otimes \mathscr{O}_{\mathbb{P}^3}(1), \qquad f_b^{+} : U_b \to U_{b-1} \otimes \mathscr{O}_{\mathbb{P}^3}(1),$$
$$f_b^{-} : U_b \to U_{b+1} \otimes \mathscr{O}_{\mathbb{P}^3}(1), \qquad f_b^{--} : U_b \to U_{b+3} \otimes \mathscr{O}_{\mathbb{P}^3}(1).$$

According to our convention the expression of the map f^{++} takes the form

$$(f_b^{++})_{i,j} = \frac{\binom{3}{j-i}\binom{b-3}{i-1}}{\binom{b}{j-1}} x_{j-i}. \tag{3.6}$$

Similarly, for the remaining maps we have the expression

$$(f_b^{\epsilon})_{i,j} = \frac{\sum_{q=0}^{s} (-1)^q \binom{s}{q}\binom{3-q}{j-i-q+s}\binom{b+q+s-3}{i+q-s-1}}{\binom{b}{j-1}} x_{j-i+s}, \quad \text{with } \begin{cases} \epsilon = + \Rightarrow s = 1, \\ \epsilon = - \Rightarrow s = 2, \\ \epsilon = -- \Rightarrow s = 3. \end{cases} \tag{3.7}$$

We will also need the expression of the unique (up to a scalar) equivariant duality $J_b : U_b \to U_b$, which will be skew-symmetric as soon as b is odd. This takes the form

$$(J_b)_{i,j} = \frac{(-1)^i}{\binom{b}{j-1}} \delta_{i,b-j+2}, \quad \text{with } \delta_{h,k} = \begin{cases} 1 & \text{if } h = k, \\ 0 & \text{if } h \neq k. \end{cases} \tag{3.8}$$

Looking back at Lemma 3.3, we can write down explicitly the form of the equivariant map defining an $\mathsf{SL}(2)$-homogeneous instanton bundle.

Remark 3.5. Let $A : W \to I \otimes \mathscr{O}_{\mathbb{P}^3}(1)$ be a matrix defining an $\mathsf{SL}(2)$-homogeneous instanton bundle $\mathscr{E}$ on $\mathbb{P}(U_3)$. Then A takes the form

$$A = \begin{pmatrix} g^{++}_{2a+1} & g^{-}_{2a-3} & g^{--}_{2a-5} & 0 & & & & & \\ 0 & g^{++}_{2a-3} & g^{+}_{2a-5} & g^{-}_{2a-7} & g^{--}_{2a-9} & 0 & & & \\ & 0 & 0 & g^{++}_{2a-7} & g^{+}_{2a-9} & g^{-}_{2a-11} & g^{--}_{2a-13} & 0 & \\ & & & & 0 & \ddots & \ddots & \ddots & 0 \\ & & & & & 0 & g^{++}_{5} & g^{+}_{3} & g^{-}_{1} \end{pmatrix}, \tag{3.9}$$

for some even integer a, or

$$A = \begin{pmatrix} g^{++}_{2a+1} & g^{-}_{2a-3} & g^{--}_{2a-5} & 0 & & & & \\ 0 & g^{++}_{2a-3} & g^{+}_{2a-5} & g^{-}_{2a-7} & g^{--}_{2a-9} & 0 & & \\ & 0 & 0 & g^{++}_{2a-7} & g^{+}_{2a-9} & g^{-}_{2a-11} & g^{--}_{2a-13} & 0 \\ & & & 0 & \ddots & \ddots & \ddots & g^{--}_{1} \\ & & & & 0 & 0 & g^{++}_{3} & 0 \end{pmatrix}, \tag{3.10}$$

for some odd integer a, where the map g^{ϵ}_b is of the form $\mathsf{c}^{\epsilon}_b \cdot f^{\epsilon}_b$, c^{ϵ}_b lies in $\mathbb{C}$, f^{ϵ}_b is defined in Remark 3.4, and ϵ ranges in $\{++, +, -, --\}$.

Lemma 3.6. *Let A be defined by (3.9) or (3.10). Then A has maximal rank everywhere if and only if $\mathsf{c}^{++}_b \neq 0$, for each b.*

Proof. Assume $\mathsf{c}^{++}_b \neq 0$, for each b. By Lemma 3.1, in order to prove that A has maximal rank on $\mathbb{P}(U_3)$, it suffices to check that it does so on the point $p_0 = (1 : 0 : 0 : 0) \in \Gamma$. Let A_0 be the evaluation of A at p_0. By the expression (3.6) for f^{++} the j-th entry on the main diagonal of A_0 takes the form

$$\mathsf{c}^{++}_b \cdot \frac{\binom{b-3}{j-1}}{\binom{b}{j-1}},$$

which is nonzero as soon as $b \geq 4$. On the other hand we always have $b \geq 3$, while $b = 3$ implies $j = 0$. So this coefficient never vanishes. Therefore A_0 is upper triangular with nonvanishing terms on the main diagonal; hence it has maximal rank.

Conversely, assume that a coefficient c^{++}_b is zero, and consider the row of the matrix A_0 containing g^{++}_b. This row contains at most the three matrices g^{+}_{b-2}, g^{-}_{b-4}, g^{--}_{b-6}. By the expression (3.7), in each of these matrices the top row vanishes on p_0. So A_0 cannot have maximal rank. □

Lemma 3.7. *Let A be defined by (3.9), and let $J : W^* \to W$ be the equivariant duality defined by the matrix*

$$J = \begin{pmatrix} J_{2a+1} & 0 & & \\ 0 & J_{2a-3} & 0 & \\ & 0 & \ddots & 0 \\ & & 0 & J_1 \end{pmatrix},$$

where each J_b is defined by (3.8). Then the equation

$$A J A^{\top} = 0$$

is equivalent to the following system of equations:

$$\frac{\binom{2a-2}{3}}{\binom{2a+1}{3}}(\mathsf{c}^{++}_{2a+1})^2 + \frac{2a-4}{2a-3}(\mathsf{c}^{-}_{2a-3})^2 = (\mathsf{c}^{--}_{2a-5})^2, \tag{3.11}$$

$$9\frac{2a-2}{\binom{2a+1}{2}}(\mathsf{c}^{++}_{2a+1})^2 = 2(\mathsf{c}^{-}_{2a-3})^2, \tag{3.12}$$

$$\frac{9}{5}(\mathsf{c}^{++}_{5})^2 - \frac{2}{3}(\mathsf{c}^{+}_{3})^2 = 2(\mathsf{c}^{-}_{1})^2, \tag{3.13}$$

and, for each $s = 1, \ldots, a/2 - 2$,

$$\frac{\binom{2a-2-4s}{3}}{\binom{2a+1-4s}{3}}(\mathsf{c}^{++}_{2a+1-4s})^2 - \frac{\binom{2a-3-4s}{2}}{\binom{2a-1-4s}{2}}(\mathsf{c}^{+}_{2a-1-4s})^2 + \frac{2a-4-4s}{2a-3-4s}(\mathsf{c}^{-}_{2a-3-4s})^2 = (\mathsf{c}^{--}_{2a-5-4s})^2, \tag{3.14}$$

$$9\frac{2a-2-4s}{\binom{2a+1-4s}{2}}(\mathsf{c}^{++}_{2a+1-4s})^2 - 2\frac{2a-5-4s}{2a-1-4s}(\mathsf{c}^{+}_{2a-1-4s})^2 = -2(\mathsf{c}^{-}_{2a-3-4s})^2, \tag{3.15}$$

and, for each $t = 1, \ldots, a/2 - 1$,

$$\mathsf{c}^{+}_{2a-1-4t}\mathsf{c}^{--}_{2a-1-4t} = \frac{2a-2-4t}{2a+1-4t}\mathsf{c}^{++}_{2a+1-4t}\mathsf{c}^{-}_{2a+1-4t}. \tag{3.16}$$

Proof. The map $AJA^\top$ is an $\mathsf{SL}(2)$-equivariant skew-symmetric matrix with quadratic entries, which we can identify with an invariant element of $\wedge^2(\wedge^2 U_a) \otimes (U_6 \oplus U_2)$.

By decomposing I into $\mathsf{SL}(2)$-irreducible summands we obtain a block decomposition of this matrix, i.e. we write $AJA^\top = (B_{i,j})$, where the block $B_{i,j}$ represents the map

$$B_{i,j} : U_{2a-2-4i} \to U_{2a-2-4j} \otimes \mathscr{O}_{\mathbb{P}^3}(2),$$

which is induced by $AJA^\top$.

Clearly, the only nonzero blocks sit along the three central diagonals. Moreover, since the map $AJA^\top$ is skew-symmetric, we need not impose conditions on the blocks sitting above the main diagonal, as soon as the remaining blocks vanish.

Now, a block sitting on the main diagonal corresponds to the induced map $U_b \to U_b \otimes \mathscr{O}_{\mathbb{P}^3}(2)$. For $b \geq 4$, this block vanishes as soon as the coefficients of x_3^2 and x_2^2 are zero; indeed these monomials generate $U_6 \oplus U_2$ as an $\mathfrak{sl}(2)$-module. Making use of the expressions (3.6)–(3.8), one derives by a direct computation the conditions (3.11), (3.12), (3.14) and (3.15). These amount to $a - 2$ equations. On the other hand, for $b = 2$ we need only impose that the coefficient of x_2^2 be zero. This gives Eq. (3.13).

The blocks sitting on the diagonal below the main one correspond to maps $U_b \to U_{b-4} \otimes \mathscr{O}_{\mathbb{P}^3}(2)$. Notice that here we only have to take care of the coefficient of x_3^2. By a direct computation, this gives the condition (3.16), which amounts to $a/2 - 1$ equations. □

Theorem 3.8. *For each integer* $a \geq 2$, *there exists an* $\mathsf{SL}(2)$-*instanton bundle* $\mathscr{E}$ *on* $\mathbb{P}(U_3)$ *with* $c_2(\mathscr{E}) = \binom{a}{2}$. *The matrix* $A : W \to I(1)$ *representing* $\mathscr{E}$ *is unique up to the action of* $\mathsf{Sp}(W) \times \mathsf{SL}(I)$.

Proof. According to the parity of a, we have to check that there exists a matrix of the form (3.9) or (3.10) having everywhere maximal rank, and satisfying $AJA^\top = 0$. We work out the case (3.9), the other one being similar.

Consider now the matrix A. It has $a/2$ rows and a columns, and it depends only on the $2a - 2$ coefficients of the form $\mathsf{c}^{\epsilon}_{b}$. In view of Lemma 3.6, we assume $\mathsf{c}^{++}_{2a+1-4s} \neq 0$, for each $s = 0, \ldots, a/2 - 1$. Imposing nonzero values on these coefficients, we are left with $(3a)/2 - 2$ variables. On the other hand Lemma 3.7 gives $(3a)/2 - 2$ homogeneous quadratic equations. So there exists a solution and we find the matrix A.

To prove uniqueness, we look more carefully at our set of equations. Fix first the $a/2$ nonzero coefficients of the form $\mathsf{c}^{++}_{2a+1-4s}$. Observe that the Eqs. (3.11) and (3.12) determine c^{-}_{2a-3} and c^{--}_{2a-5} up to the choice of some sign. Then (3.16) for $t = 1$ gives a unique value for c^{+}_{2a-5}. Now (3.15) and (3.14) for $s = 1$ give c^{-}_{2a-7} and c^{--}_{2a-9} up to some sign. Then again we use (3.16), (3.15) and (3.14) until we are left with (3.16) for $t = a/2 - 1$. After that we use (3.13) to settle c^{-}_{1} up to sign. So the only choice for A is the choice of the $\mathsf{c}^{++}_{2a+1-4s}$'s and of the sign in the solutions of the system of equation of Lemma 3.7.

Now making use of the $\mathsf{SL}(I)$-action via diagonal matrices, we may multiply the rows of A by any nonzero scalar. Recall that the remaining coefficients depend linearly on the $\mathsf{c}^{++}_{2a+1-4s}$'s besides the choice of signs, so we may assume that the $\mathsf{c}^{++}_{2a+1-4s}$'s are all equal to 1.

Finally we use the $\mathsf{Sp}(W)$-action. We make use of diagonal transformations consisting of blocks of $+1$'s and -1's, each of the size of some $J_b : U_b \to U_b$ in the $\mathsf{SL}(2)$-decomposition of W. These transformations allow us to change the sign in the columns of A, so we may indifferently pick any sign in the choice of the solution of (3.11)–(3.15), as long as the ratio

$$\frac{\mathsf{c}^{-}_{2a+1-4t}\mathsf{c}^{++}_{2a+1-4t}}{\mathsf{c}^{--}_{2a-1-4t}\mathsf{c}^{+}_{2a-1-4t}}$$

remains unchanged for $t = 1, \dots, a/2 - 1$. But the Eq. (3.16) prescribes that this be equal to $\frac{2a+1-4t}{2a-1-4t}$. □

3.3. The resolution an $\mathsf{SL}(2)$-instanton bundle on $\mathbb{P}(U_3)$

Here we provide another, much simpler way to define the instanton bundle $\mathscr{E}$ described above. Let $b \le 1$ be an integer, and set $h_b = f^{-}_{b}$.

Lemma 3.9. *For $b \ge 1$, the sheaf* $\operatorname{coker}(h_b)$ *is locally free of rank 2 if and only if b is divisible by 3.*

Proof. Consider the points $p_1 = (0 : 1 : 1 : 0)$, $p_0 = (1 : 0 : 0 : 0)$ in $\mathbb{P}(U_3)$. Notice that p_1 sits in $\mathbb{P}(U_3) \setminus Y_4$ while p_0 sits in Γ.

We can depict the matrix h_b as follows:

$$h_b = \begin{pmatrix} -x_2 & \alpha^{(b)}_{1,2}x_3 & 0 & & & \\ -2x_1 & \alpha^{(b)}_{2,2}x_2 & \alpha^{(b)}_{2,3}x_3 & 0 & & \\ x_0 & \alpha^{(b)}_{3,2}x_1 & \alpha^{(b)}_{3,3}x_2 & \alpha^{(b)}_{3,4}x_3 & 0 & \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & 0 & \alpha^{(b)}_{1,2}x_0 & -x_1 \end{pmatrix}.$$

The first two rows of h_b vanish at p_0, so clearly the map h_b has corank 2 at p_0 for each b if the coefficient $\alpha^{(b)}_{j+2,j}$ is nonzero for each b. This is indeed the case for we have

$$\alpha^{(b)}_{j+2,j} = \frac{b+1-j}{b}.$$

Now consider the matrix H_b obtained by evaluating h_b at p_1. In view of Lemma 3.1, we have to check that H_b has corank 2 if and only if b is divisible by 3. This holds true if we check that all the coefficients $\alpha^{(b)}_{j+1,j}$ multiplying x_1 in h_b are nonzero, except one of them when b is divisible by 3. Making use of (3.7), an easy computation leads to the formula

$$\alpha^{(b)}_{j+1,j} = \frac{3\binom{b-1}{j-2} - 2\binom{b}{j-1}}{\binom{b}{j-1}}.$$

So we have

$$\alpha^{(b)}_{j+1,j} = 0 \iff \begin{cases} b = 3d; \\ j = 2d+1. \end{cases}$$

Our claim is thus proved. □

Lemma 3.10. *Let $d \geq 1$ and set $\mathscr{E} = \operatorname{coker}(h_{3d})(-d)$. Then $\mathscr{E}$ is stable with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = \binom{d+1}{2}$. We have an equivariant exact sequence:*

$$0 \to \mathscr{O}(-2d-1) \to U_{3d} \otimes \mathscr{O}_{\mathbb{P}^3}(-d-1) \to U_{3d+1} \otimes \mathscr{O}_{\mathbb{P}^3}(-d) \to \mathscr{E} \to 0, \tag{3.17}$$

where each map is the unique (up to scalar) equivariant morphism between the given source and target.

Proof. By the previous lemma, $\ker h_{3d}$ is a line bundle. So let $\ker h_{3d} = \mathscr{O}_{\mathbb{P}^3}(-\ell)$. Since the map $\mathscr{O}_{\mathbb{P}^3}(-\ell) \to U_{3d} \otimes \mathscr{O}_{\mathbb{P}^3}(-d-1)$ is equivariant, there must be an invariant element in $\operatorname{Hom}_{\mathbb{P}^3}(\mathscr{O}_{\mathbb{P}^3}(-\ell), U_{3d} \otimes \mathscr{O}_{\mathbb{P}^3}(-d-1)) \cong \operatorname{Sym}^{\ell-d-1} U_3 \otimes U_{3d}$. This implies $\ell \geq 2d+1$.

On the other hand, consider the unique (up to scalar) equivariant map $k_d : \mathscr{O}_{\mathbb{P}^3}(-2d-1) \to U_{3d} \otimes \mathscr{O}_{\mathbb{P}^3}(-d-1)$. The composition $h_{3d} \circ k_d$ is again equivariant. But the representation $\operatorname{Hom}_{\mathbb{P}^3}(\mathscr{O}_{\mathbb{P}^3}(-2d-1), U_{3d+1} \otimes \mathscr{O}_{\mathbb{P}^3}(-d))$ is isomorphic to $\operatorname{Sym}^{d+1} U_3 \otimes U_{3d+1}$, which contains no invariant element; hence we get $h_{3d} \circ k_d = 0$. Then the line bundle $\mathscr{O}_{\mathbb{P}^3}(-2d-1)$ sits in $\ker h_{3d}$, so $\ell \leq 2d+1$.

This gives the exact sequence (3.17). It follows that $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = \binom{d+1}{2}$. Since $\mathscr{E}$ has no global sections, it is stable by Hoppe's criterion. □

The following proposition is due to Giorgio Ottaviani.

Proposition 3.11. *For any $d \geq 1$ the natural composition*

$$\operatorname{Sym}^{d-1}(U_3) \otimes U_{3d} \to \operatorname{Sym}^{d-1}(U_3) \otimes \operatorname{Sym}^{d}(U_3) \to \operatorname{Sym}^{2d-1}(U_3)$$

is surjective.

Proof. By Hermite reciprocity we reduce to prove the surjectivity of the natural map:

$$\operatorname{Sym}^3(U_{d-1}) \otimes U_{3d} \xrightarrow{\phi} \operatorname{Sym}^3(U_{2d-1}).$$

Choose now a monomial order such that $\mathsf{y} < \mathsf{x}$. Once we have fixed an integer b, we use the notation $y_n = \mathsf{x}^{b-n}\mathsf{y}^n$ for a basis of U_b. Note that we have an induced order on the basis of U_b, that is $y_b < y_{b-1} < \cdots < y_0$.

A basis of $\operatorname{Sym}^3(U_b)$ is given by the symmetric product $y_{b-n_3} y_{b-n_2} y_{b-n_1}$. where $n_1 \geq n_2 \geq n_3$. We consider the lexicographic order on the basis of $\operatorname{Sym}^3(U_b)$; that is

$$y_{b-n_3} y_{b-n_2} y_{b-n_1} < y_{b-m_3} y_{b-m_2} y_{b-m_1}$$

if the first nonzero entry in $(m_1 - n_1, m_2 - n_2, m_3 - n_3)$ is positive. Hence we have

$$y_b^3 < \cdots < y_0^2 y_1 < y_0^3.$$

Note that, under the embedding: $U_{3b} \hookrightarrow \operatorname{Sym}^3(U_b)$, the image of the our basis can be written as follows:

$$y_b^3 < y_{b-1} y_b^2 < y_{b-2} y_b^2 + y_{b-1}^2 y_b < \cdots < y_0^2 y_2 + y_0 y_1^2 < y_0^2 y_1 < y_0^3.$$

Here we may assume without loss of generality that the coefficients are all equal to one. Note that $y_{b-2} y_b^2$ is the leading term of $y_{b-2} y_b^2 + y_{b-1}^2 y_b$.

Set now $b = 2d-1$ and consider the space $\operatorname{Sym}^3(U_{2d-1})$. We have thus $y_n = \mathsf{x}^{2d-1-n}\mathsf{y}^n$. Define also the basis elements of U_{d-1} and U_{3d} as follows:

$$u_n = \mathsf{x}^{d-1-n}\mathsf{y}^n, \qquad v_n = \mathsf{x}^{3d-n}\mathsf{y}^n.$$

We claim that every basis element

$$\mathsf{B}_{n_1,n_2,n_3} := y_{2d-n_3} y_{2d-n_2} y_{2d-n_1},$$

with $n_1 \geq n_2 \geq n_3$, belongs to the image of ϕ. We prove this claim by induction on the order just defined. There are four cases to be considered.

(i) If $n_1 \leq d-1$, then we have

$$\mathsf{B}_{n_1,n_2,n_3} = \phi\left(u_{d-1-n_3}u_{d-1-n_2}u_{d-1-n_1} \otimes v_{3d}\right),$$

where the subcase $n_i = 0$ is the starting point of the inductive argument.

(ii) If $n_2 \leq d-1 \leq d \leq n_1$, then the image

$$\phi(u_0 u_{d-1-n_3}u_{d-1-n_2} \otimes v_{4d-1-n_1} + \cdots)$$

equals B_{n_1,n_2,n_3} plus lower terms which belong to the image of ϕ by the inductive assumption. It follows that in this case B_{n_1,n_2,n_3} belongs to the image of ϕ.

(iii) If $n_3 \leq d-1 \leq d \leq n_2$, then

$$\phi(u_{2d-n_2-1}u_{2d-n_1-1}u_{d-1} \otimes v_{d-n_3} + \cdots),$$

is equal to B_{n_1,n_2,n_3} plus lower terms which again lie in the image of ϕ by induction. Thus B_{n_1,n_2,n_3} sits in $\mathrm{Im}(\phi)$.

(iv) Finally, in the case of $a \leq n_3$, we obtain

$$\mathsf{B}_{n_1,n_2,n_3} = \phi(u_{2d-1-n_3}u_{2d-1-n_2}u_{2d-1-n_1} \otimes v_0). \quad \square$$

Corollary 3.12. *For $d \geq 1$, the bundle $\mathscr{E}$ on $\mathbb{P}(U_3)$ defined by the exact sequence (3.17) is an instanton bundle.*

Proof. By the Lemmas 3.9 and 3.10, it suffices to show that $\mathrm{H}^1(\mathbb{P}^3, \mathscr{E}(-2)) = 0$. We get the equality

$$\mathrm{H}^1(\mathbb{P}^3, \mathscr{E}(-2)) = \ker(\mathrm{H}^3(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(-2d-3)) \to \mathrm{H}^3(\mathbb{P}^3, U_{3d} \otimes \mathscr{O}_{\mathbb{P}^3}(-d-3))).$$

By Serre duality this gives

$$\mathrm{H}^1(\mathbb{P}^3, \mathscr{E}(-2))^* = \mathrm{coker}(\mathrm{H}^0(\mathbb{P}^3, U_{3d} \otimes \mathscr{O}_{\mathbb{P}^3}(d-1)) \xrightarrow{\phi'} \mathrm{H}^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2d-1))).$$

The map ϕ' identifies with ϕ of the previous proposition; hence we are done. $\square$

Remark 3.13. In the cases $a = 2, 3$, the $\mathsf{SL}(2)$-homogeneous instanton bundle of the previous corollary was first constructed by P. Katsylo and G. Ottaviani during the preparation of [11] by computational tools. L. Gruson observed that the case $a = 2$ has a different interpretation in terms of nets of quadrics as in [7]. It corresponds to the net of quadrics containing a twisted cubic.

Remark 3.14. Set $k = \binom{d+1}{2}$. For $a = 1, 2$, changing the maps in Eq. (3.17), we obtain the minimal graded free resolution of a general k-instanton. For higher d this is no longer true, for the general k-instanton has no sections at the twist d; see [9].

For $d \leq 6$ (that is, for $k \leq 21$) a proof of smoothness of $\mathsf{MI}(k)$ at an instanton bundle $\mathscr{E}$ given by Theorem 3.8 can be achieved by making use of `Macaulay2` [6]. Namely, for $d \leq 6$ we write down the matrix h_{3d} and set $\mathscr{E} = \mathrm{coker}(h_{3d})$. Then the `Macaulay2` computation, performed over a finite field, gives $\mathrm{H}^2(\mathbb{P}^3, \mathscr{E} \otimes \mathscr{E}) = 0$. This implies our claim.

Acknowledgments

I am grateful to Giorgio Ottaviani for many valuable ideas, and for generously sharing with me his results. In particular Proposition 3.11 and Corollary 3.12 are due to him. I would like to thank the referee for some useful remarks, in particular for pointing out a gap in the references.

References

[1] Michael F. Atiyah, Nigel J. Hitchin, Vladimir G. Drinfel'd, Yuri I. Manin, Construction of instantons, Phys. Lett. A 65 (3) (1978) 185–187. MR MR598562 (82g:81049).

[2] Wolf Barth, Klaus Hulek, Monads and moduli of vector bundles, Manuscripta Math. 25 (4) (1978) 323–347. MR MR509589 (80f:14005).

[3] Iustin Coandă, Alexander Tikhomirov, Günther Trautmann, Irreducibility and smoothness of the moduli space of mathematical 5-instantons over $\mathbb{P}_3$, Internat. J. Math. 14 (1) (2003) 1–45. MR MR1955508 (2004b:14018).

[4] Laura Costa, Giorgio Ottaviani, Group action on instanton bundles over $\mathbb{P}^3$, Math. Nachr. 246–247 (1) (2002) 31–46.

[5] Laura Costa, Giorgio Ottaviani, Nondegenerate multidimensional matrices and instanton bundles, Trans. Amer. Math. Soc. 355 (1) (2003) 49–55.
[6] Daniel R. Grayson, Michael E. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/.
[7] Laurent Gruson, Mohamed Skiti, 3-instantons et réseaux de quadriques, Math. Ann. 298 (2) (1994) 253–273. MR MR1256618 (94m:14012).
[8] Robin Hartshorne, Stable vector bundles of rank 2 on $\mathbf{P}^3$, Math. Ann. 238 (3) (1978) 229–280. MR MR514430 (80c:14011).
[9] Robin Hartshorne, André Hirschowitz, Cohomology of a general instanton bundle, Ann. Sci. École Norm. Sup. (4) 15 (2) (1982) 365–390. MR MR683638 (84c:14011).
[10] André Hirschowitz, Mudumbai S. Narasimhan, Fibrés de 't Hooft spéciaux et applications, in: Enumerative Geometry and Classical Algebraic Geometry (Nice, 1981), in: Progr. Math., vol. 24, Birkhäuser, Boston, MA, 1982, pp. 143–164. MR MR685768 (84g:14016).
[11] Pavel I. Katsylo, Giorgio Ottaviani, Regularity of the moduli space of instanton bundles $\mathrm{MI}_{\mathbf{P}^3}(5)$, Transform. Groups 8 (2) (2003) 147–158. MR MR1976457 (2004f:14057).
[12] Christian Okonek, Michael Schneider, Heinz Spindler, Vector bundles on complex projective spaces, in: Progress in Mathematics, vol. 3, Birkhäuser, Boston, MA, 1980. MR 81b:14001.
[13] Giorgio Ottaviani, Günther Trautmann, The tangent space at a special symplectic instanton bundle on $\mathbf{P}_{2n+1}$, Manuscripta Math. 85 (1) (1994) 97–107. MR MR1299051 (95k:14014).
[14] Heinz Spindler, Günther Trautmann, Special instanton bundles on $\mathbf{P}_{2N+1}$, their geometry and their moduli, Math. Ann. 286 (1–3) (1990) 559–592. MR MR1032947 (91c:14056).
[15] Jean Vallès, Fibrés de Schwarzenberger et coniques de droites sauteuses, Bull. Soc. math. France 128 (3) (2000) 433–449.